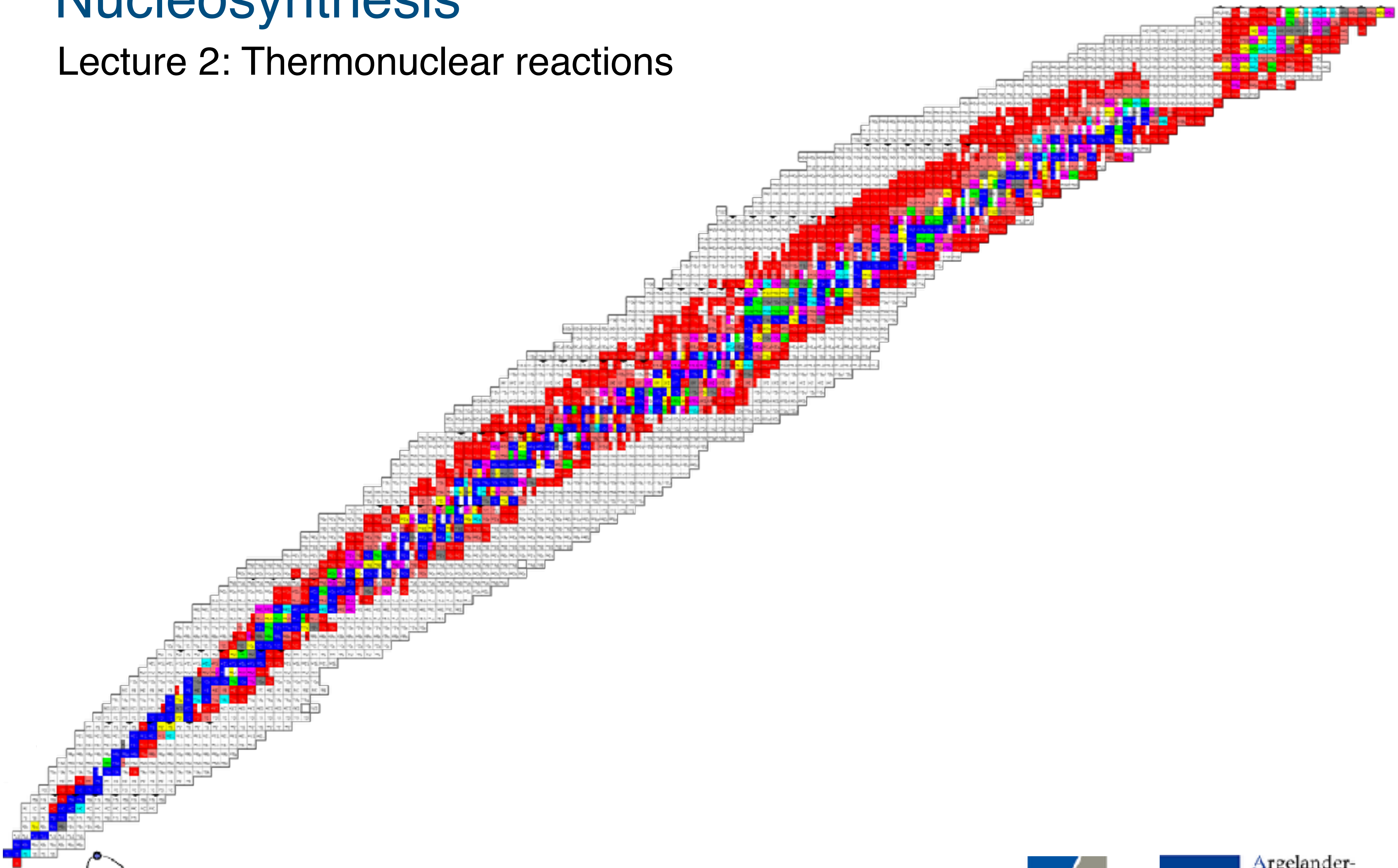
Nucleosynthesis

Lecture 2: Thermonuclear reactions



Bonn-Cologne Graduate School
of Physics and Astronomy

UNIVERSITÄT BONN

Argelander-
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Astronomie

Overview

- **Lecture 1:** Introduction & overview **April 18**
- **Lecture 2:** Thermonuclear reactions **April 25**
- **Lecture 3:** Big-bang nucleosynthesis **May 2**
- **Lecture 4:** Thermonuclear reactions inside stars — I (H-burning) **May 7**
- **Lecture 5:** Thermonuclear reactions inside stars — II (advanced burning) **May 16**
- **Lecture 6:** Neutron-capture and supernovae — I **May 23**
- **Lecture 7:** Neutron-capture and supernovae — II **June 6**
- **Lecture 8:** Thermonuclear supernovae **June 13**
- **Lecture 9:** Li, Be and B **July 4**
- **Lecture 10:** Galactic chemical evolution and relation to astrobiology **July 11**

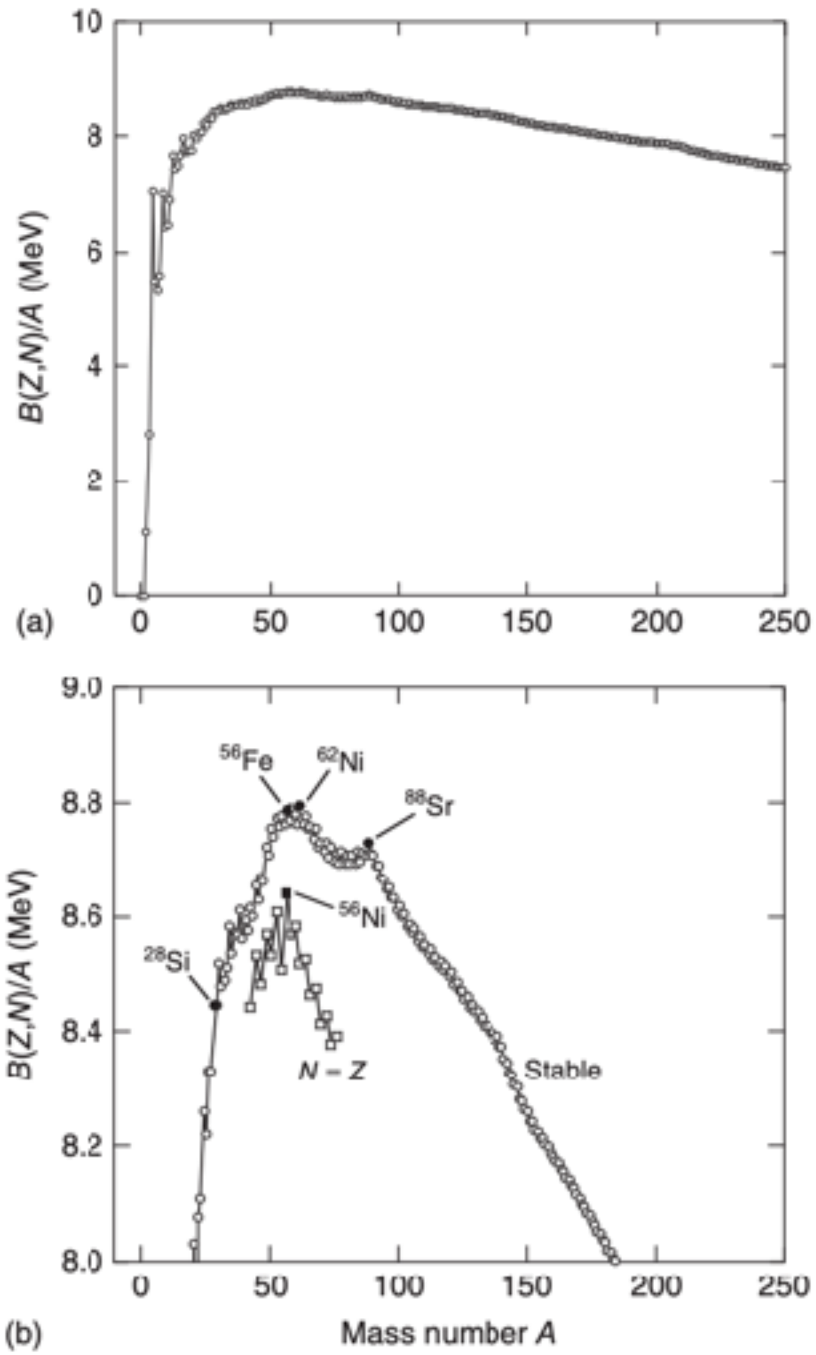
Paper presentations I **June 21**

Paper presentations II **June 27**

Why does the Sun shine?

Answer: the Sun shines *because it is massive*

However, it can shine for billions of years only if nuclear reactions take place in its interior



Solar luminosity:

$$L_\odot = 3.9 \times 10^{33}\,\mathrm{ergs\,s^{-1}}$$

Total rest-mass energy of the Sun

$$M_\odot c^2 \simeq 2 \times 10^{54}\,\mathrm{ergs}$$

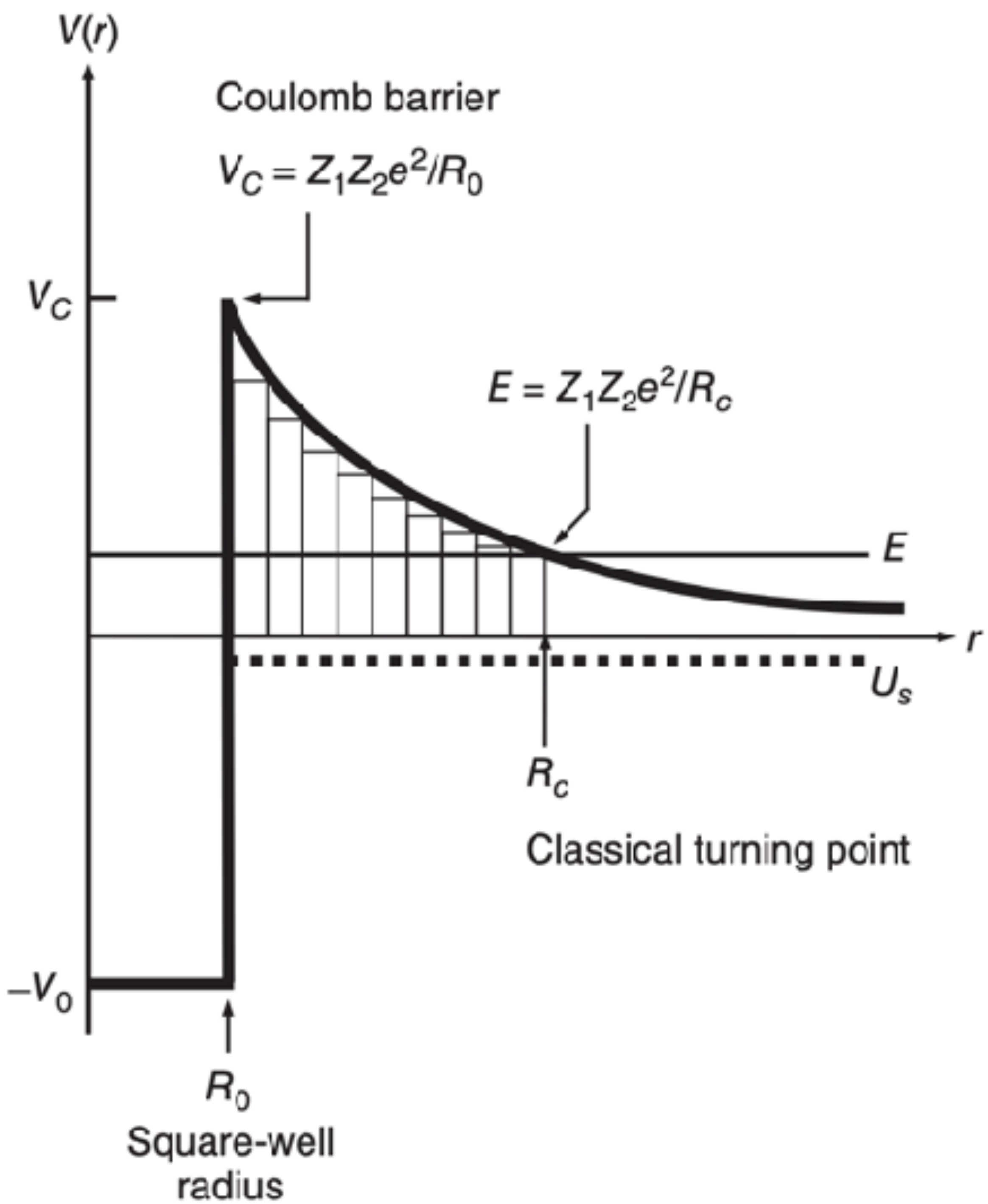
If the entire mass is converted to energy at present luminosity then the Sun can keep on shining for

$$M_\odot c^2 / L_\odot \simeq 1.6 \times 10^{13}\,\mathrm{years}$$

However, this is not realistic, since only a fraction of the mass can be converted to energy.

Since different elements have different binding energies, fusion is the most likely energy source

Occurrence of nuclear reactions



Characteristic length-scale of nuclear forces

$$R_0 \simeq A^{1/3} 1.44\,\mathrm{fm}$$

Reactions can occur "classically" when

$$E_{\mathrm{particle}} \geq V(R_0) \simeq Z_1 Z_2\,\mathrm{MeV}$$

For the centre of the Sun:

$$E_{\mathrm{particle}} = kT_{\mathrm{c}} \simeq k10^7\,\mathrm{K} \simeq 1\,\mathrm{keV} \ll 1\,\mathrm{MeV}$$

Probability for "classical" interaction
$\exp(-1000) \simeq 10^{-434}$

Reactions take place at relatively low temperatures because of the "tunnelling effect" discovered by Gamow.
Allows penetration at radii larger than classical capture radius.

Capture probability:

$$P_0 \propto E^{-1/2} \exp(-2\pi\eta); \eta = \frac{\mu}{2} \frac{Z_1 Z_2 e^2}{\hbar E^{1/2}}$$

Probability for "quantum" interaction
$P_0 \simeq 10^{-20}$

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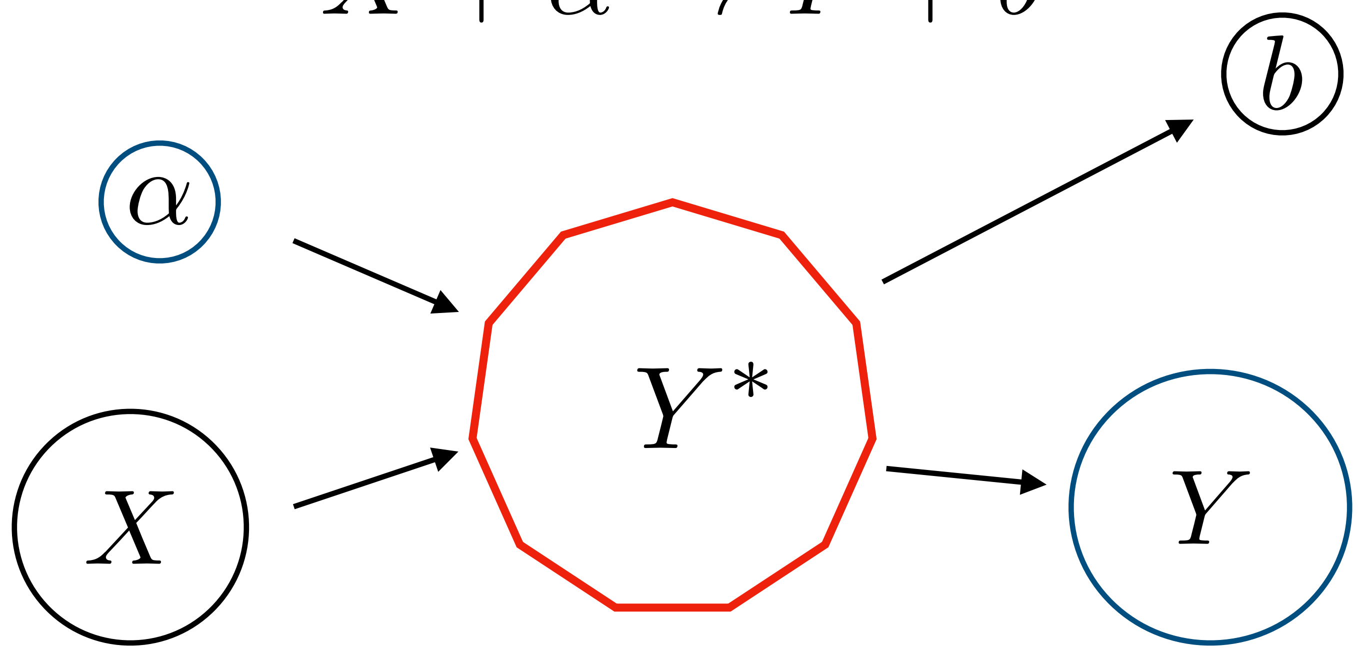
Energetics of nuclear reactions

$$X + \alpha \to Y + b$$



Alternative notation: $X(\alpha, b)Y$

Conservation of relativistic energy

$$E_{\alpha X} + (M_\alpha + M_X)c^2 = E_{Yb} + (M_b + M_Y)c^2$$

Conservation of charge; nuclear masses can be replaced by atomic masses

$$E_{\alpha X} + (M_{\alpha'} + M_{X'})c^2 = E_{Yb} + (M_{b'} + M_{Y'})c^2$$

Conservation of number of nuclei

$$M_{\alpha'}c^2 \to \Delta M_{\alpha'}c^2 = (M_{\alpha'} - A \times 1\,\mathrm{amu})c^2$$

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Energetics of nuclear reactions

$$\boxed{\begin{aligned} Q &\equiv E_{Y'b'} - E_{\alpha' X'} \\ &= (\Delta M_{\alpha'} + \Delta M_{X'} - \Delta M_{Y'} - \Delta M_{b'})c^2 \end{aligned}}$$

Exothermic reaction $Q > 0$

Endothermic reaction $Q < 0$

Binding energy: energy required to disassemble the nucleus to its constituent neutrons and protons

$$\text{B.E.} = [m_p \times Z + m_n \times (A - Z) - M_X]c^2$$

Table 1.1 Experimental values of the atomic mass excess (M.E.), binding energy per nucleon (B/A), and relative atomic mass (M) for light nuclides in the $A \leq 20$ mass region.

A	Element	M.E. (keV)	B/A (keV)	M (u)
1	n	8071.3171	0.0	1.0086649158
	H	7288.97059	0.0	1.00782503223
2	H	13135.72174	1112.283	2.01410177812
3	H	14949.8061	2827.266	3.0160492779
	He	14931.2155	2572.681	3.0160293201
4	He	2424.91561	7073.915	4.00260325413
6	Li	14086.8789	5332.331	6.0151228874
7	Li	14907.105	5606.439	7.016003437
	Be	15769.00	5371.548	7.01692872
8	Li	20945.80	5159.712	8.02248625
	Be	4941.67	7062.435	8.00530510
	B	22921.6	4717.15	8.0246073
9	Li	24954.90	5037.768	9.02679019
	Be	11348.45	6462.668	9.01218307
10	Be	12607.49	6497.630	10.01353470
	B	12050.7	6475.07	10.0129369
11	Be	20177.17	5952.540	11.02166108
	B	8667.9	6927.72	11.0093054
	C	10650.3	6676.37	11.0114336
12	B	13369.4	6631.22	12.0143527
	C	0.0	7680.144	12.0000000
13	B	16562.1	6496.41	13.0177802
	C	3125.00875	7469.849	13.00335483507
	N	5345.48	7238.863	13.00573861
14	C	3019.893	7520.319	14.003241988
	N	2863.41669	7475.614	14.00307400443
	O	8007.46	7052.301	14.00859636
15	C	9873.1	7100.17	15.0105993
	N	101.4387	7699.460	15.0001088989
	O	2855.6	7463.69	15.0030656
16	N	5683.9	7373.80	16.0061019
	O	−4737.00137	7976.206	15.99491461957
17	N	7870.0	7286.2	17.008449
	O	−808.7636	7750.728	16.9991317565
	F	1951.70	7542.328	17.00209524
18	N	13113.0	7038.6	18.014078
	O	−782.8156	7767.097	17.9991596129
	F	873.1	7631.638	18.0009373
19	O	3332.9	7566.49	19.0035780
	F	−1487.4443	7779.018	18.9984031627
	Ne	1752.05	7567.342	19.00188091
20	F	−17.463	7720.134	19.99998125
	Ne	−7041.9306	8032.240	19.9924401762
	Na	6850.6	7298.50	20.0073544

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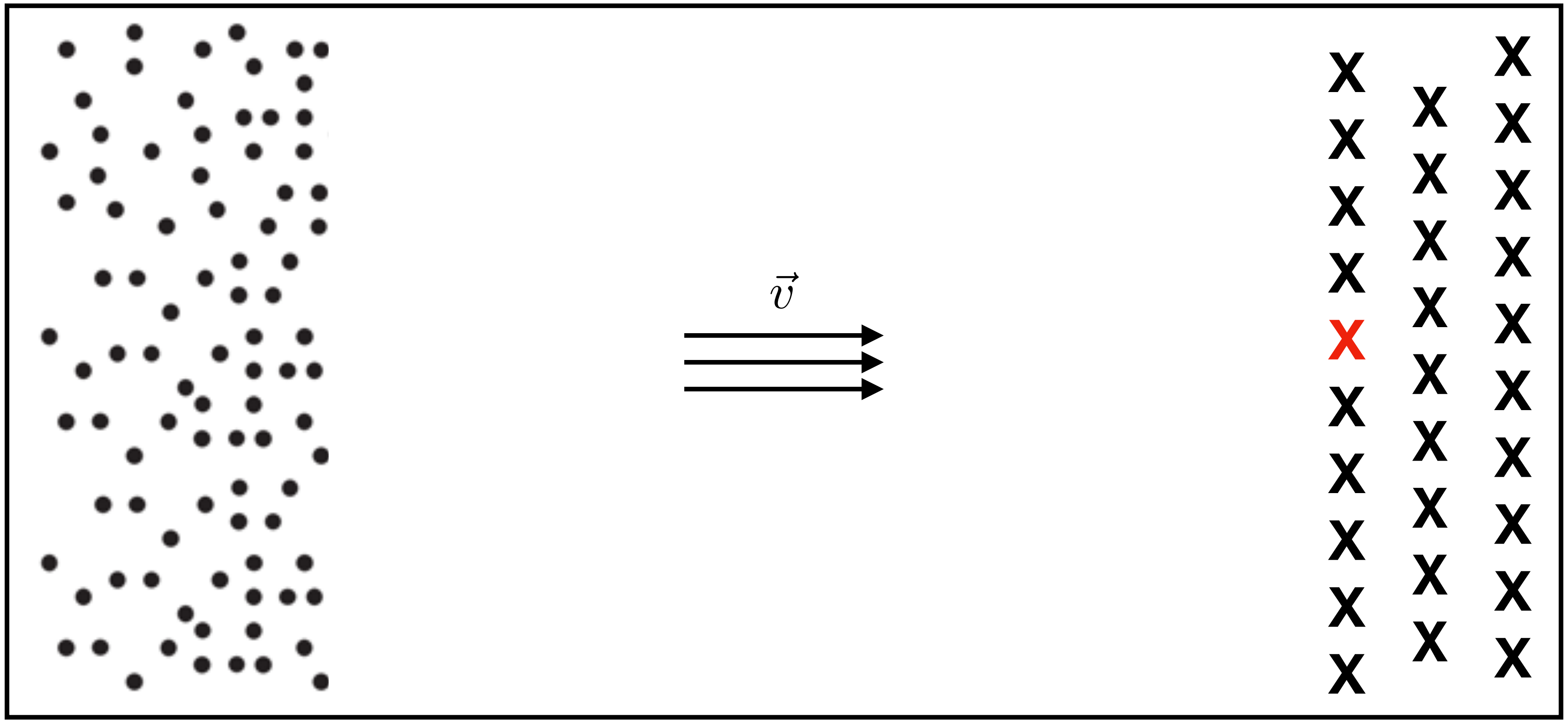
Illiadis (2015); Wang et al. (2012)

Rate of nuclear reactions

$$X(\alpha, \beta)Y$$



$$\text{cross section} \equiv \frac{\text{number of reactions/target nucleus/unit time}}{\text{number of incident particles/unit area/unit time}}$$

- The cross section, σ of a reaction, as defined here, has units of area, and can be thought of as the area around a nucleus X, over which an incoming particle can be captured (although this picture is not accurate).
- σ can vary with velocity, i.e. energy

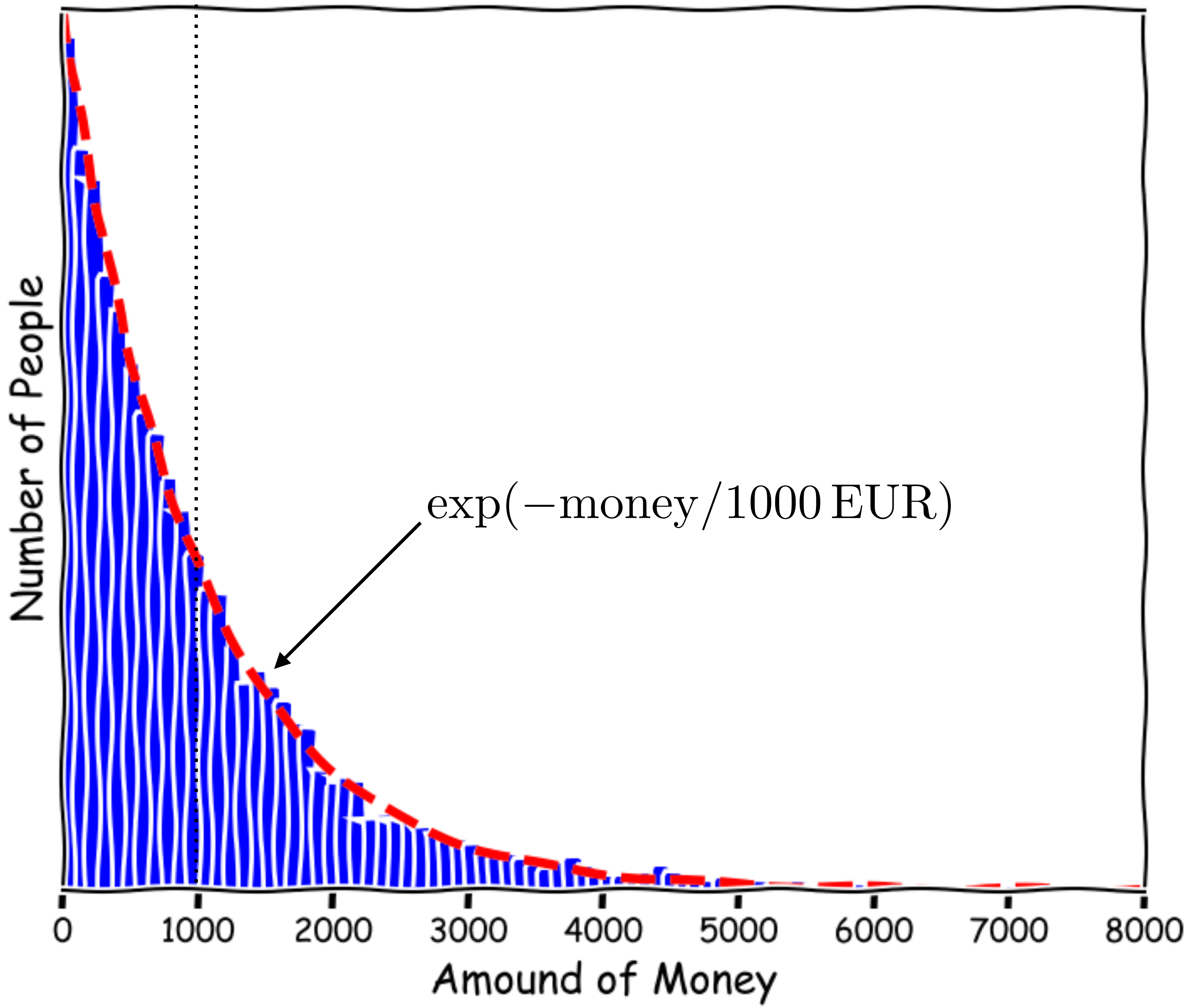
From definition: $\dfrac{\text{number of reactions}}{\text{unit time unit volume}} = \text{cross section} \times \text{velocity} \times N_\alpha N_X \Rightarrow r_{\alpha X} = \sigma_{\alpha X}(v) v N_\alpha N_X$

Rate of nuclear reactions

strong force:	$^{15}\mathrm{N}(\mathrm{p},\alpha)^{12}\mathrm{C}$	$\sigma \simeq 0.5$ b at $E_p = 2$ MeV
el. mag. force:	$^{3}\mathrm{He}(\alpha,\gamma)^{7}\mathrm{Be}$	$\sigma \simeq 10^{-6}$ b at $E_\alpha = 2$ MeV
weak force:	$\mathrm{p}(\mathrm{p},\mathrm{e}^{+}\nu)\mathrm{D}$	$\sigma \simeq 10^{-20}$ b at $E_p = 2$ MeV

Thermal equilibrium

Give each person in a group 1000€ and let them exchange freely. The total amount of money is conserved



Rate of nuclear reactions

In a mixture of gases in thermal equilibrium, there exists a spectrum of *relative velocities* between the particles of type α and X, just as α and X *individually* have well defined velocity spectra

$$r_{\alpha X} = N_\alpha N_X \int_0^\infty v\sigma(v)\phi(v)dv = N_\alpha N_X \langle \sigma v \rangle \qquad \text{with} \int \phi(v)dv = 1$$

More generally, $r_{\alpha X} = (1+\delta_{\alpha X})^{-1} N_a N_X \langle \sigma v \rangle$
so the entire *problem reduces to determining <σv>*

Nuclei inside stars, with the exception of neutron stars, are always non-degenerate, therefore the spectrum of relative velocities follows the Maxwell-Boltzmann distribution

$$\boxed{r_{\alpha X} = (1+\delta_{\alpha X})^{-1} N_\alpha N_X 4\pi \left(\frac{\mu}{2\pi kT}\right)^{3/2} \int_0^\infty v^3 \sigma(v) \exp\left(-\frac{\mu v^2}{2kT}\right) dv}$$

Lifetime of nucleus X

$$\left(\frac{\partial N_X}{\partial t}\right)_\alpha \equiv -\frac{N_X}{\tau_\alpha(X)}$$

Combined with the definition for the nuclear reaction rate, it follows that

$$\tau_\alpha(X) = (1+\delta)^{-1}\frac{N_X}{r_{\alpha X}} = (\langle \sigma v \rangle N_\alpha)^{-1}$$ for multiple reactions involving X: $$\frac{1}{\tau(X)} = \sum \frac{1}{\tau_i(X)}$$

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Rate of nuclear reactions

Evaluation of the integral

$$r_{\alpha X} = (1+\delta_{\alpha X})^{-1} N_\alpha N_X 4\pi \left(\frac{\mu}{2\pi kT}\right)^{3/2} \int_0^\infty v^3 \sigma(v) \exp\left(-\frac{\mu v^2}{2kT}\right) dv$$

Step 1: Evaluate $\sigma(v)$

Depends on quantum interactions, coulomb barrier, nuclear properties…impossible to evaluate analytically, but we can do some tricks

$$\sigma(v) = \underbrace{\text{Penetration Probability}}_{\exp\left(-\frac{2\pi Z_1 Z_2 e^2}{\hbar u}\right)} \times \underbrace{\text{Size (De Broglie)}}_{\frac{1}{E}} \times \underbrace{\text{Nuclear Properties} \times \text{constant factor}}_{\text{Astrophysical Factor} := S(E)}$$

i. e. we can “hide” all unknown nuclear terms and constants inside S(E)

$$\sigma(E) \equiv S(E) E^{-1} \exp(-bE^{-1/2}); b = 31.28 Z_1 Z_2 A^{1/2}$$

The energy difference between nuclear states is usually larger than the energy range of nuclear reactions inside stars.
This means that in many cases S(E) is a slowly varying function of energy and can be fixed to a constant when evaluating astrophysical rates

Rate of nuclear reactions

Evaluation of the integral

$$r_{\alpha X} = (1+\delta_{\alpha X})^{-1} N_\alpha N_X 4\pi \left(\frac{\mu}{2\pi kT}\right)^{3/2} \int_0^\infty v^3 \sigma(v) \exp\left(-\frac{\mu v^2}{2kT}\right) dv$$

Step 1: Evaluate $\sigma(v)$

$\sigma(v)$ can vary by many orders of magnitude over a relatively narrow energy range

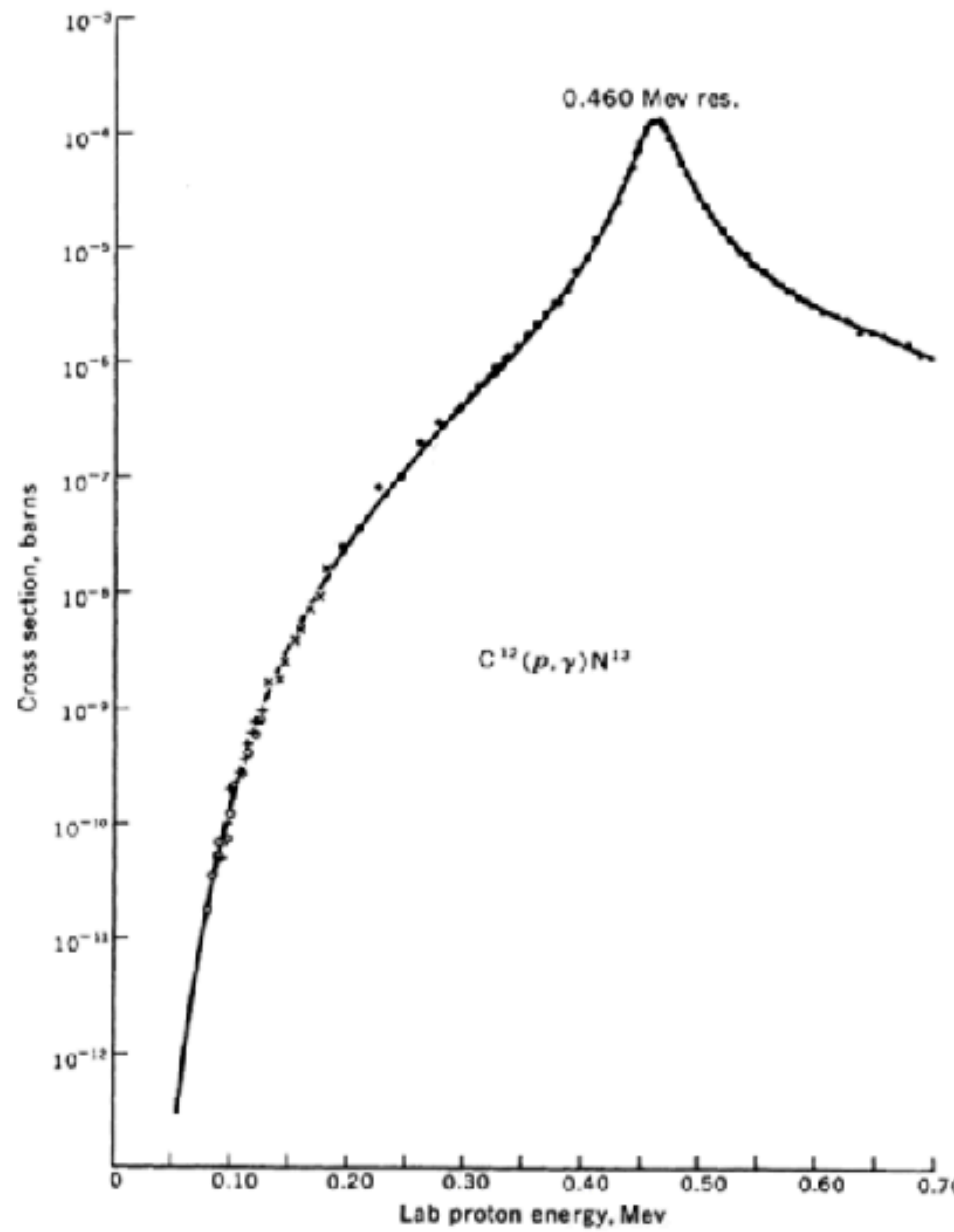
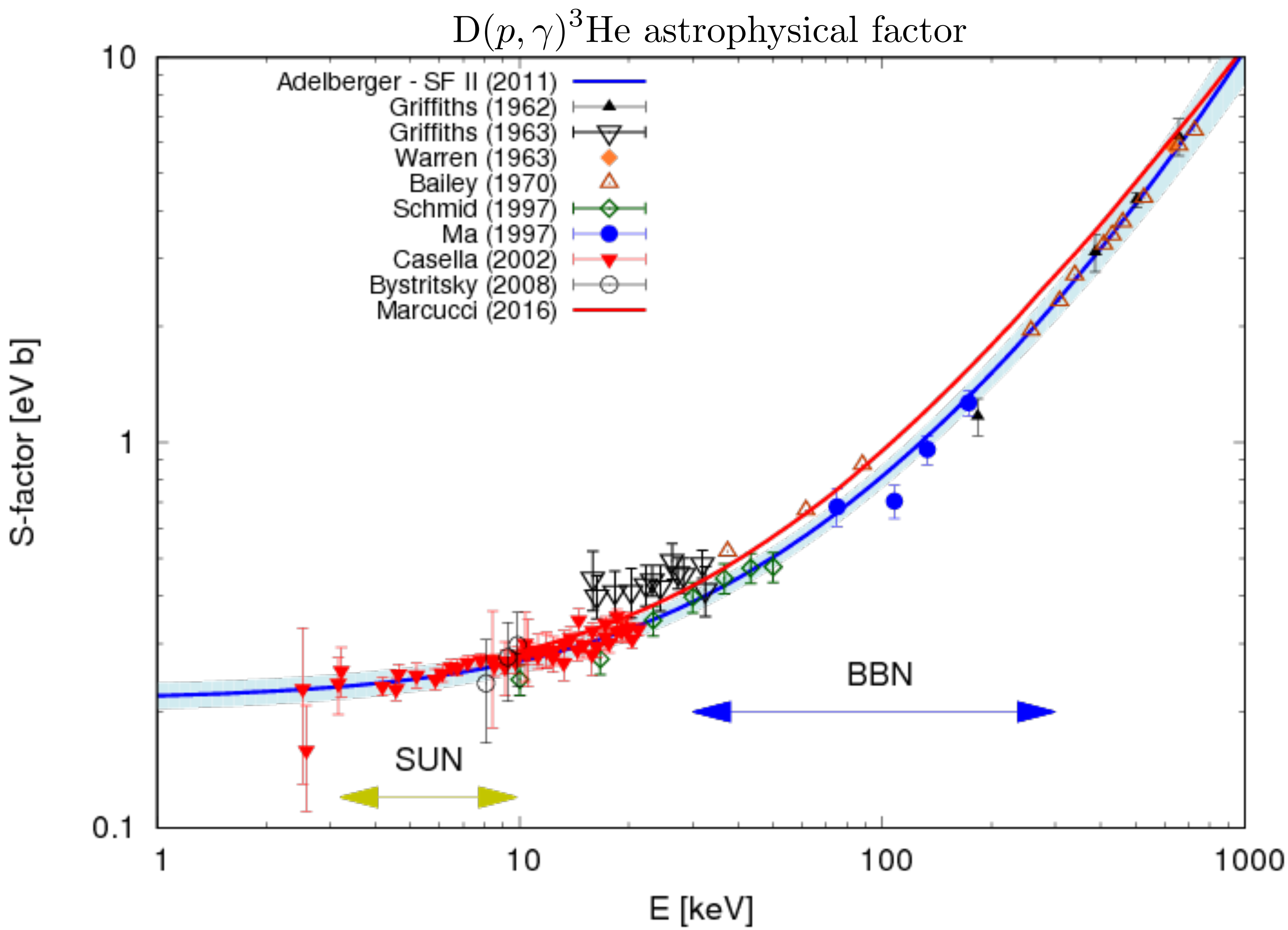


Figure by CLAYTON

Rate of nuclear reactions

Often S(E) is measured only at high energies in the lab and extrapolated down to astrophysical energies. Only recently direct measurements at low energies are becoming possible (still for a limited number of reactions)



Rate of nuclear reactions

Evaluation of the integral

$$r_{\alpha X} = (1+\delta_{\alpha X})^{-1} N_\alpha N_X 4\pi \left(\frac{\mu}{2\pi kT}\right)^{3/2} \int_0^\infty v^3 \sigma(v) \exp\left(-\frac{\mu v^2}{2kT}\right) dv$$

Step 2: Replace $\sigma(v)$; $v \to E$

$$r_{\alpha X} = \frac{N_\alpha N_X}{1+\delta_{\alpha X}} 4\pi \left(\frac{\mu}{2\pi kT}\right)^{3/2} \overbrace{\boxed{\left(\frac{8}{\mu\pi}\right)^{1/2} \frac{1}{(kT)^{3/2}} \int_0^\infty S(E) \exp\left(-\frac{E}{kT} - bE^{-1/3}\right) dE}}^{\langle \sigma v \rangle}$$

Step 3: For non-resonant reactions, assume $S(E) \simeq const = S_0$

$$J = \int_0^\infty \exp(-E/kT - bE^{-1/2}) dE$$

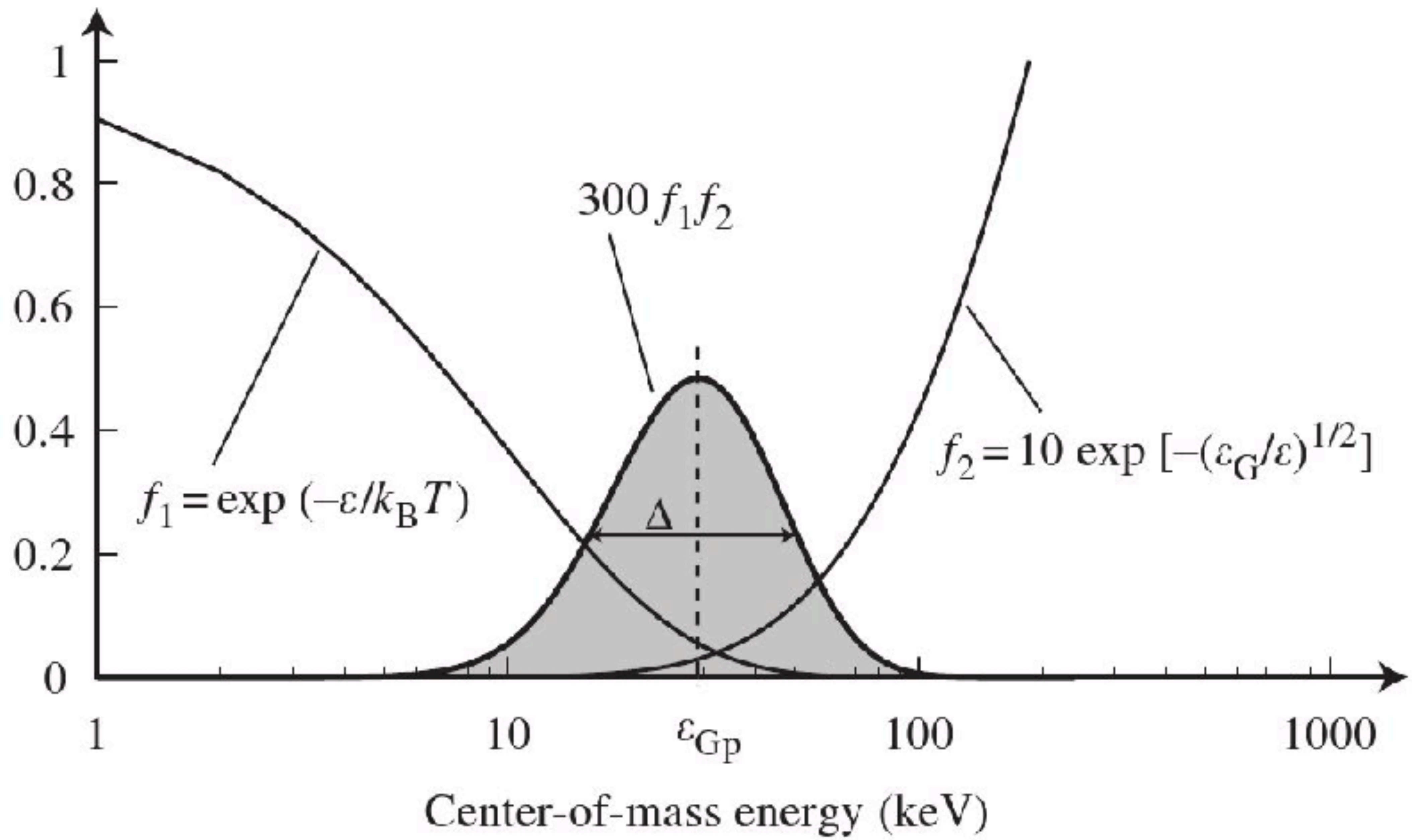
The integrand is significant only around a small range of energies ΔE around maximum, f(E)' = 0

$$E_0 = (0.5bkT)^{2/3} = \left[\left(\frac{\mu}{2}\right)^{1/2} \pi \frac{Z_1 Z_k e^2 kT}{\hbar}\right]^{2/3}$$

Rate of nuclear reactions

The Gamow Peak

$$E_0 = (0.5bkT)^{2/3} = \left[\left(\frac{\mu}{2} \right)^{1/2} \pi \frac{Z_1 Z_k e^2 kT}{\hbar} \right]^{2/3}$$



Rate of nuclear reactions

Evaluation of the integral

$$r_{\alpha X} = (1+\delta_{\alpha X})^{-1} N_\alpha N_X 4\pi \left(\frac{\mu}{2\pi kT}\right)^{3/2} \int_0^\infty v^3 \sigma(v) \exp\left(-\frac{\mu v^2}{2kT}\right) dv$$

Step 4 (final!): Approximate Gamow peak with a Gaussian

$$\tau = 3\frac{E_0}{kT}; f(E) = f_0 + f_0'(E-E_0) + 0.5 f_0''(E/E_0)^2 + \ldots = -\tau - 0.25\tau(E/E_0-1)^2 + \ldots$$

$$\xi = (E/E_0 - 1)\sqrt{\tau}/2 \Rightarrow J \simeq \frac{2/3}{k} T\tau^{1/2} e^{-\tau} \int_{-\infty}^{\infty} e^{-xi} d\xi$$

$$J \simeq \frac{2}{3} kT\pi^{1/2}\tau^{1/2} e^{-\tau}$$

$$\langle\sigma v\rangle \simeq \frac{4}{3}\left(\frac{2}{\mu}\right)^{1/2} \frac{1}{(kT)^{1/2}} S_0 \tau^{1/2} e^{-\tau}$$

$$r_{\alpha X} \simeq \frac{2.62\times 10^{29}(A_\alpha + A_X)}{(1+\delta_{\alpha X}) Z_\alpha Z_X} \rho^2 \frac{X_\alpha X_X}{(A_\alpha A_X)^2} S_0(\mathrm{keV\ barn}) \tau^2 e^r; r = 42.48\left(\frac{Z_\alpha^2 Z_X^2 A_\alpha A_X}{T/(10^6\,\mathrm{K})(A_\alpha + A_X)}\right)^{1/3}$$

Rate of nuclear reactions

$$r_{\alpha X} \simeq \frac{2.62 \times 10^{29}(A_\alpha + A_X)}{(1+\delta_{\alpha X})Z_\alpha Z_X}\rho^2 \frac{X_\alpha X_X}{(A_\alpha A_X)^2} S_0(\mathrm{keV\,barn})\tau^2 e^r; r = 42.48\left(\frac{Z_\alpha^2 Z_X^2 A_\alpha A_X}{T/(10^6\,\mathrm{K})(A_\alpha + A_X)}\right)^{1/3}$$

Some Important properties

Effective width of Gamow peak: $\Delta = \frac{4}{\sqrt{3}}(E_0 kT)^{1/2}$

Hight of Gamow peak: $I_{\max} = \exp(-3E_0/kT)$

Rates drop enormously with increasing Coulomb-barrier

$(T = 1.5 \cdot 10^7 K)$	$\Delta/2$	$\Delta \cdot I_{\max}$
$\mathrm{p} + \mathrm{p}$	3.2 keV	$7.0 \cdot 10^{-6}$ keV
$\mathrm{p} + {}^{14}\mathrm{N}$	6.8 keV	$2.5 \cdot 10^{-26}$ keV
$\alpha + {}^{12}\mathrm{C}$	9.8 keV	$5.9 \cdot 10^{-56}$ keV
${}^{16}\mathrm{O} + {}^{16}\mathrm{O}$	20.2 keV	$2.5 \cdot 10^{-237}$ keV

Temperature dependence of nuclear reactions $\langle\sigma v\rangle = \langle\sigma v\rangle_0 \left(\frac{T}{T_0}\right)^\nu$, $\nu = \frac{\partial \ln\langle\sigma v\rangle}{\partial \ln T} = \frac{\tau}{3} - \frac{2}{3}$

$(T = 1.5 \cdot 10^7 K)$	$\langle\sigma v\rangle \sim$	E_C
$\mathrm{p} + \mathrm{p}$	$T^{3.9}$	0.55 MeV
$\mathrm{p} + {}^{14}\mathrm{N}$	T^{20}	2.27 MeV
$\alpha + {}^{12}\mathrm{C}$	T^{42}	3.43 MeV
${}^{16}\mathrm{O} + {}^{16}\mathrm{O}$	T^{182}	14.07 MeV

- **the thermonuclear reaction rate is one of the most strongly varying function treated in physics!**
- **need of efficient thermostat for regulation**
- **burning stages in stars are well separated, only few elements interact at a given time**

Rate of nuclear reactions

Errors introduced from assumptions

1. S_0 is not constant; a more accurate treatment would be to assume that it's slowly changing around Gamow peak

$$S(E) = S(E_0) + \left(\frac{dS}{dE}\right)_{E_0} (E - E_0)$$

Fix: error is reduced by replacing $S_0 = S(E_0) \rightarrow S(E_0 + \frac{5}{6}kT)$

2. Error introduced by replacing area with a gaussian function: correction factor depends on τ, $F(\tau)$

3. Interaction potential is not a pure two-body central potential. Most important modification comes from electron shielding which can lower the effective potential and increase the rate. For reactions inside stars, free electrons cluster around nuclei is spherical shells with radius equal to Debye-Hückel radius

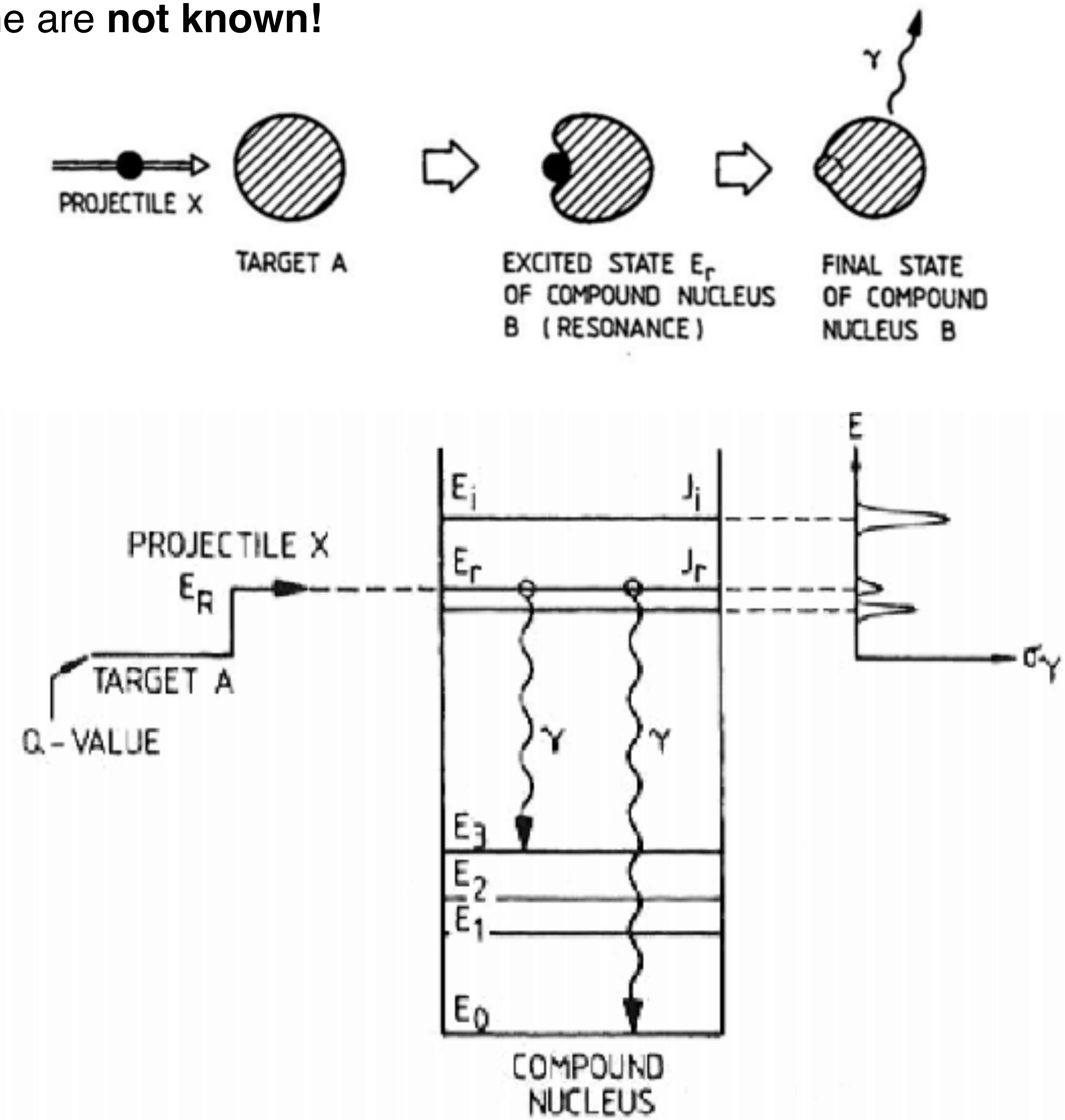
$$R_D = \left(\frac{kT}{4\pi e^2 \rho N_A \zeta}\right)^{1/2} \quad with\ \zeta = \sum_i (Z_i^2 + Z_i)\frac{Xi}{Ai}$$

Important for low temperatures and high densities!

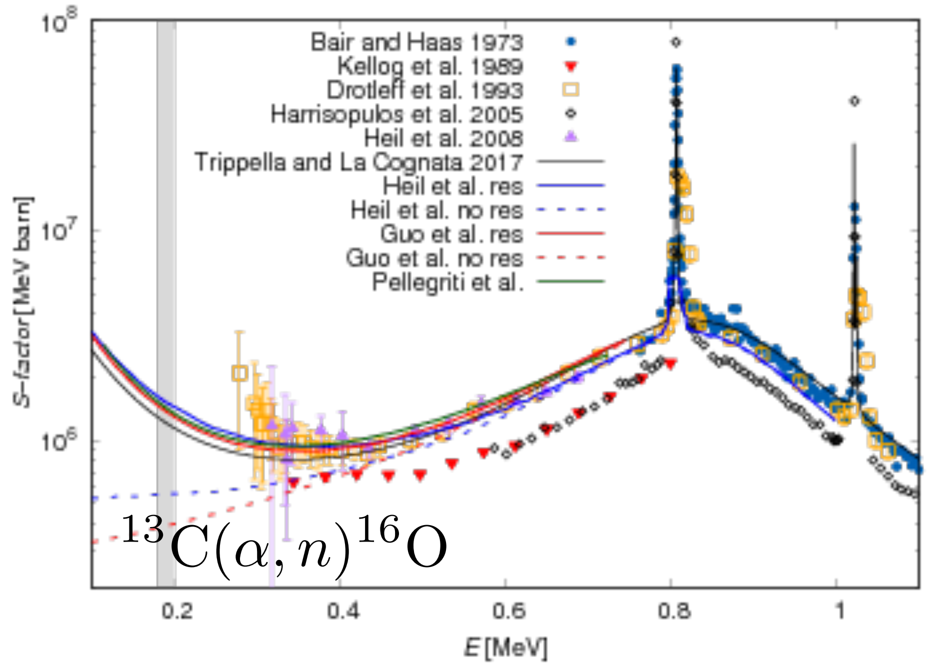
see CLAYTON for details

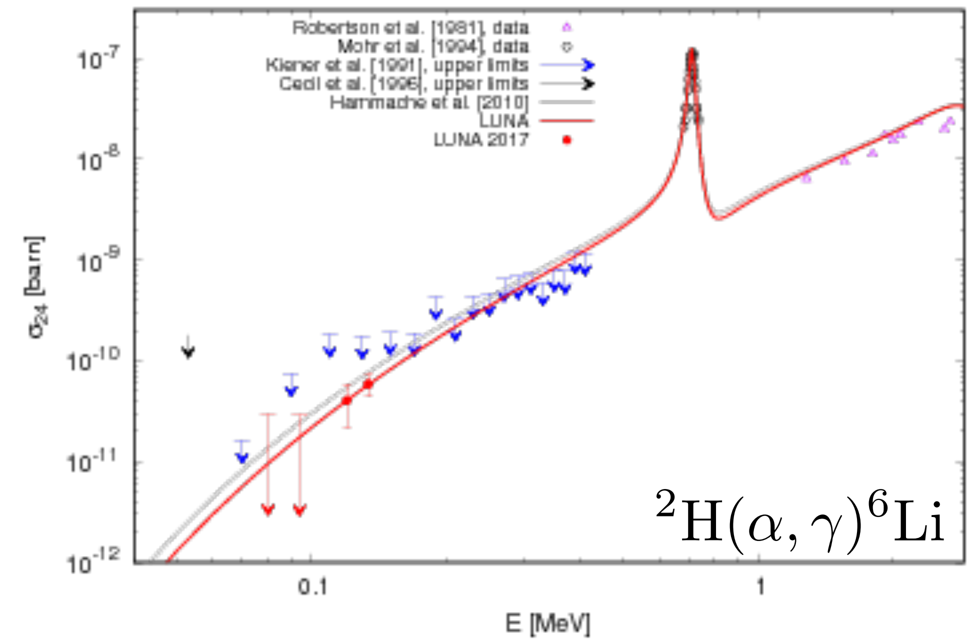
Rate of nuclear reactions

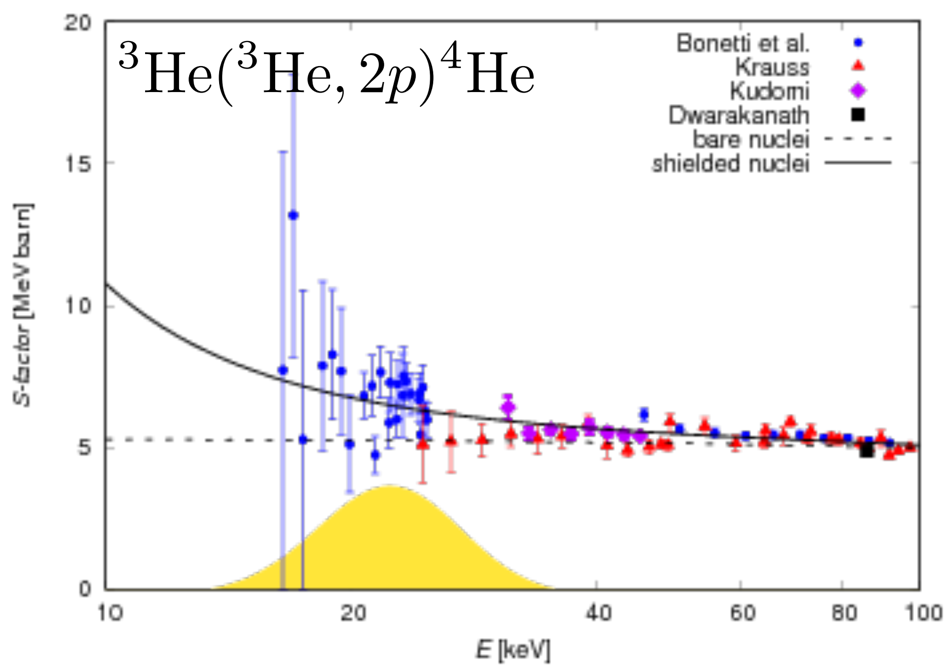
4. Nuclear resonances can introduce strong local maxima that dominate in the reaction rate. Some resonances are predicted by theory (e.g. nuclear-shell model) while others are inferred experimentally, but some are **not known!**

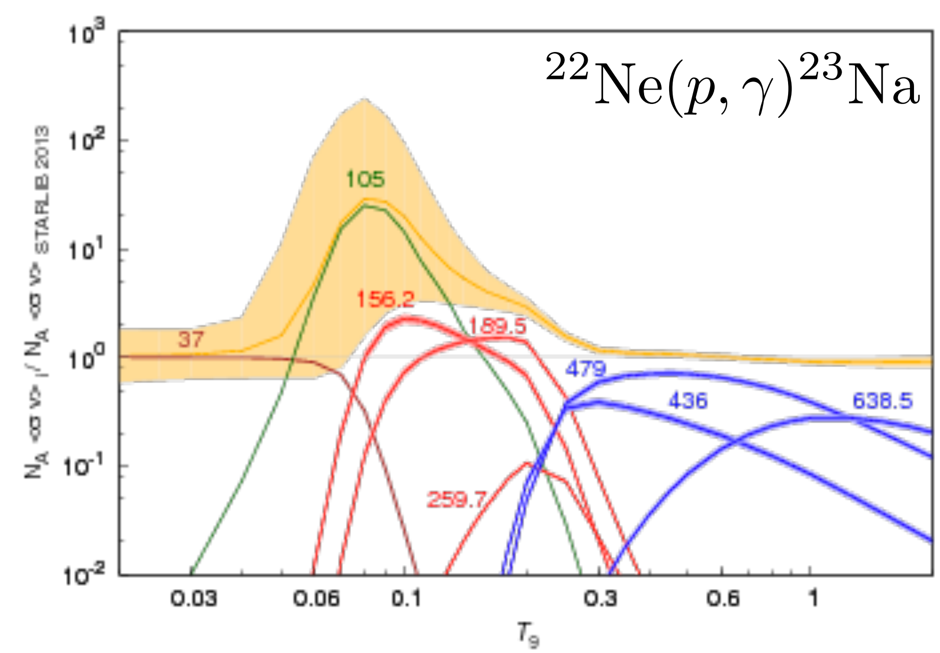


Rate of nuclear reactions









LUNA: Status and Prospects - Broggini, C. *et al.* Prog.Part.Nucl.Phys. 98 (2018) 55-84 arXiv:1707.07952

Energy generation rate

Once all reaction rates are known, the abundance changes due to nuclear burning can be calculated

$$\frac{dX_i}{dt} = A_i \frac{m_u}{\rho} \left(- \sum r_{\text{reactions that destroy } i} + \sum r_{\text{reactions that create } i} \right)$$

Energy generation rate:

$$\epsilon_{\alpha X} = \frac{Q_{\alpha X}}{m_u^2} \frac{X_\alpha X_X}{A_\alpha A_X} \rho \langle \sigma v \rangle_{\alpha X} \simeq \epsilon_0 X_\alpha X_X \rho T^\nu$$

Summary

Take Home Message

1. Thermonuclear fusion generates energy at high temperatures, providing an efficient energy source for the Sun and converting light elements to heavier (up to iron)
2. Reactions only possible due to QM tunnelling
3. Cross-section of a reaction depends on the nature of the interaction
4. Astrophysical factor often constant within energy range relevant to stars
5. Rates depend strongly on Coulomb barrier, extremely temperature-dependent
6. Burning stages inside stars are well separated in temperature
7. Rate modified by electron cloud, nuclear resonances
8. Resonances not often known

Extra Slides

The nuclear shell model

Inspired by atomic model in which the Schrödinger equation is solved in a spherically-symmetric coulomb potential, yielding discrete electron orbits (shells) characterised by certain quantum numbers for the energy (n), angular momentum (l) and spin (s).
Rules for building up the shells follow from Pauli's exclusion principle for fermions: no more than two electrons can occupy a given (n,l) state.

For nucleus, there are several complications: two types of particles (neutrons, protons), nature of the interaction is not known, interacting particles closely packed…

Nuclear shell model: Assume that each nucleon moves in a static potential

Usual assumption: Empirical Wood-Saxon central potential + strong spin-orbit coupling term (non-EM)

$$V(r) = -V_0/[1 + exp[(r - R)/a]] + V(\vec{l} \times \vec{s})$$

Solving the Schrödinger equation in 3D yields the allowed wave functions and energy levels for each nucleon

The nuclear shell model

Properties of the solution

Each nucleon is characterised by an energy level **n(>0),** an angular momentum **l(>0),** a spin **s**, and total angular momentum **j**

These combine via j-j coupling to form the total spin of the nucleus, **I**.

Parity

$$\prod = (-1)^{\sum l}$$

Energy splitting proportional to $\langle \vec{l}\ \vec{s} \rangle = 0.5(2l+1)$

Shells are filled according to Pauli's exclusion principle, for each nucleon type

$$l = 1, 2, 3, 4, 5 \ldots : s, p, d, f, g \ldots$$

$$j = l \pm s$$

Number of identical particles that can occupy a state: **2j+1**

Nuclear shell model

Important Properties

Nuclei with filled major shells of protons or neutrons (or both), exhibit an energetically favourable configuration

Magic Numbers: masses, particle separation, energies, quadrupole moments

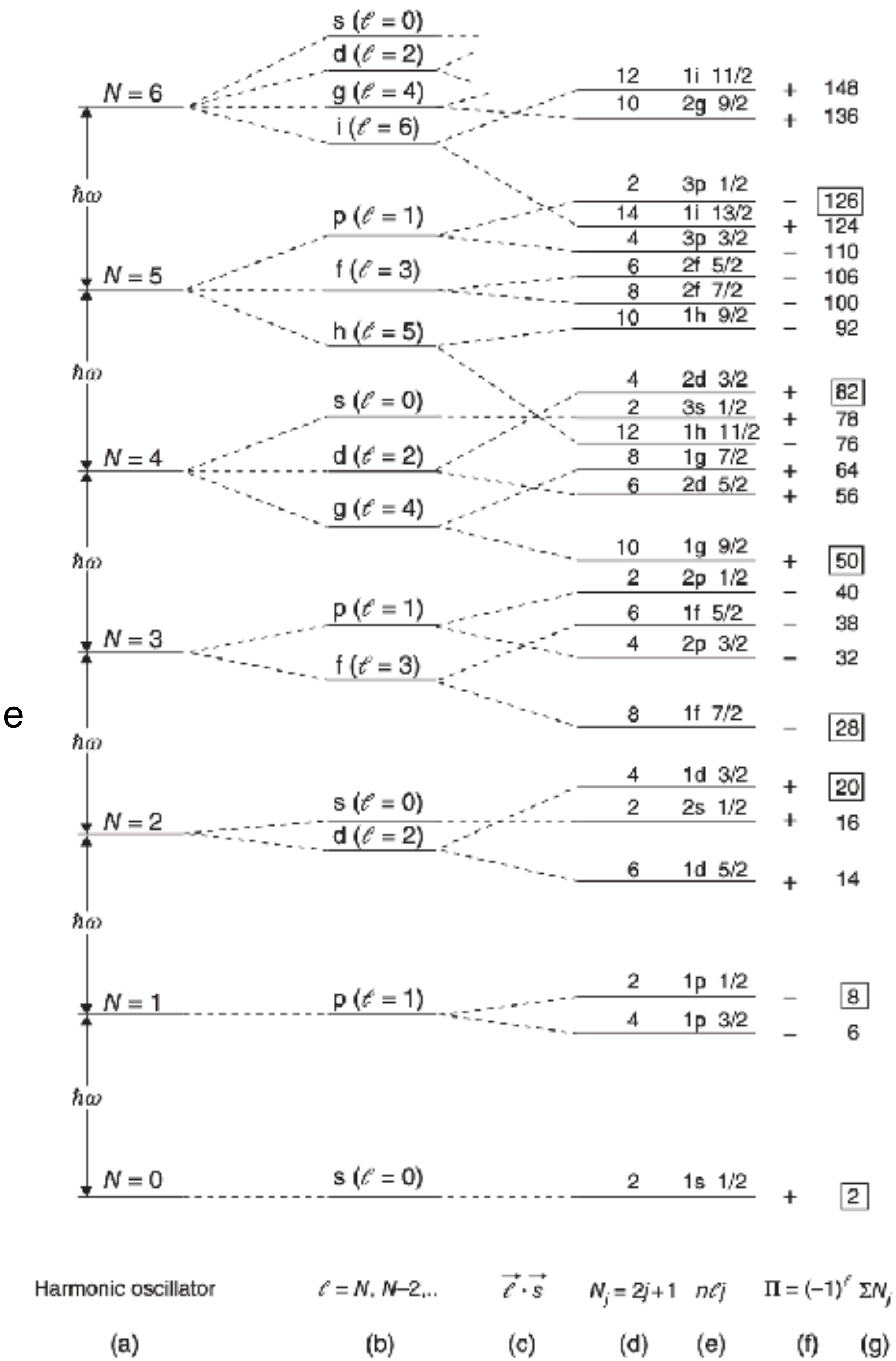
Nuclei with completely filled sub shells have I=0, Π=1

For a single nucleon outside a closed shell, ground-state spin and parity are determined by the lowest-energy state available to the nucleon
Same holds for a single “hole” in a sub-shell

Partially-filled shells are more complicated. Experimentally, the ground states of doubly even nuclei have J=0, Π=1

Energetically favourable for pairs of p,n to couple to J=0

Predictions of model, not always correct



For further reading: CLAYTON; ILIADIS

Reactions in statistical equilibrium

At very high temperatures (>5e9 K) nuclear statistical equilibrium (NSE) is achieved, i.e. the rates of all strong and electroweak reactions are exactly balanced by their reverse reactions, $dX_i/dt = 0$

$$\alpha + X \leftrightarrow b + Y$$

In NSE abundances are determined by a "nuclear Saha equation" which depends only on nuclear properties, the temperature T and the density ρ

$$X_i(A_i, Z_i, T, \rho) = \frac{A}{N_A \rho} G(A, Z, T) \left(\frac{2\pi k T M(A_i, Z_i)}{h^2} \right)^{3/2} \exp \left[\frac{\mu(A_i, Z_i) + B(A_i, Z_i)}{kT} \right]$$

where A_i is the atomic number, Z_i is the charge, T is the temperature, ρ is the mass density, N_A is the Avogadro number, $G(A, Z, T)$ is the temperature-dependent partition function, $M(A_i, Z_i)$ is the mass of the nucleus, $B(A_i, Z_i)$ is the binding energy and $\mu(A_i, Z_i)$ is the chemical potential

$$\mu(A_i, Z_i) = Z_i \mu_p + N_i \mu_n = Z_i \mu_p + (A_i - Z_i) \mu_n$$

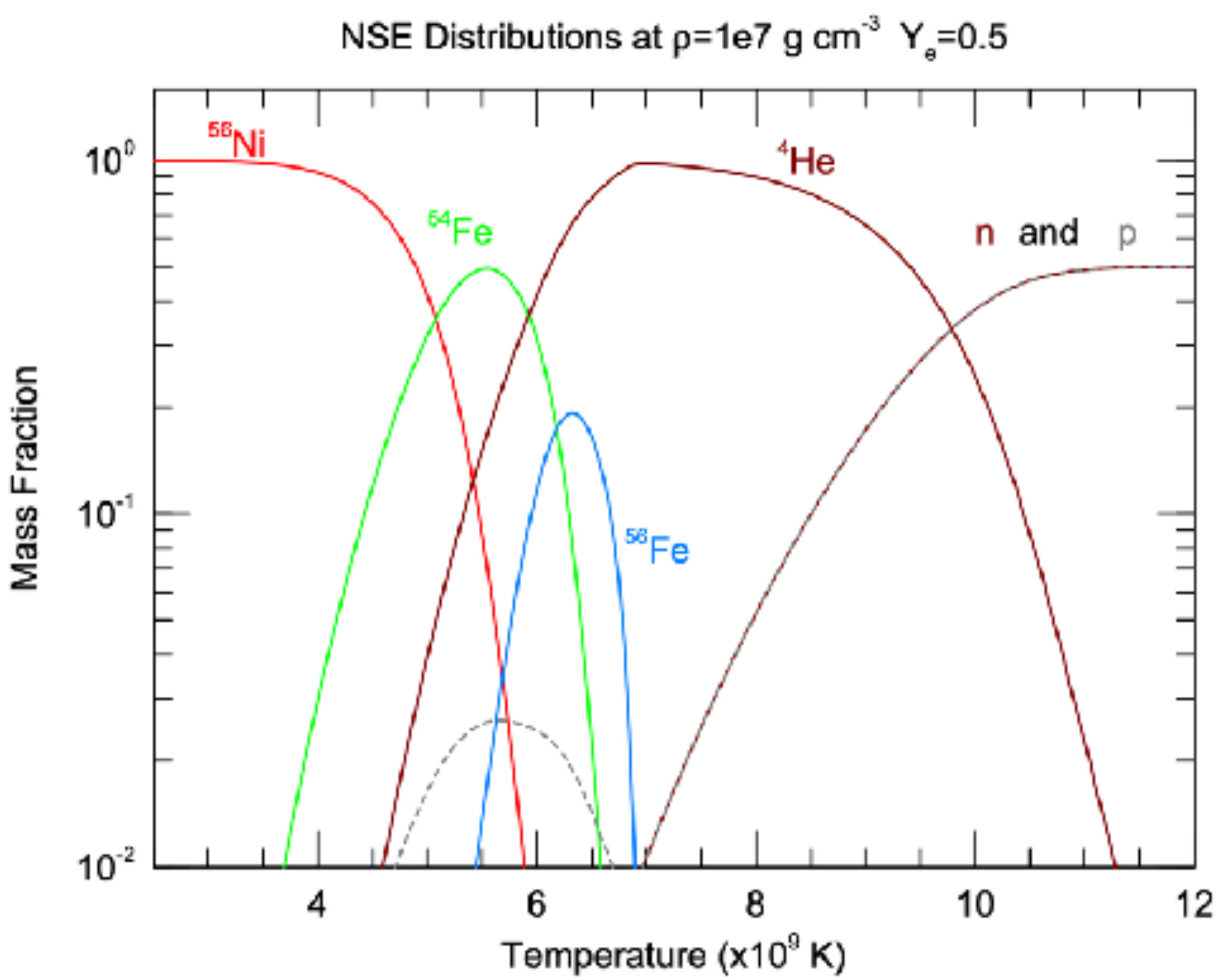
Considering conservation of baryon number, and conservation of charge

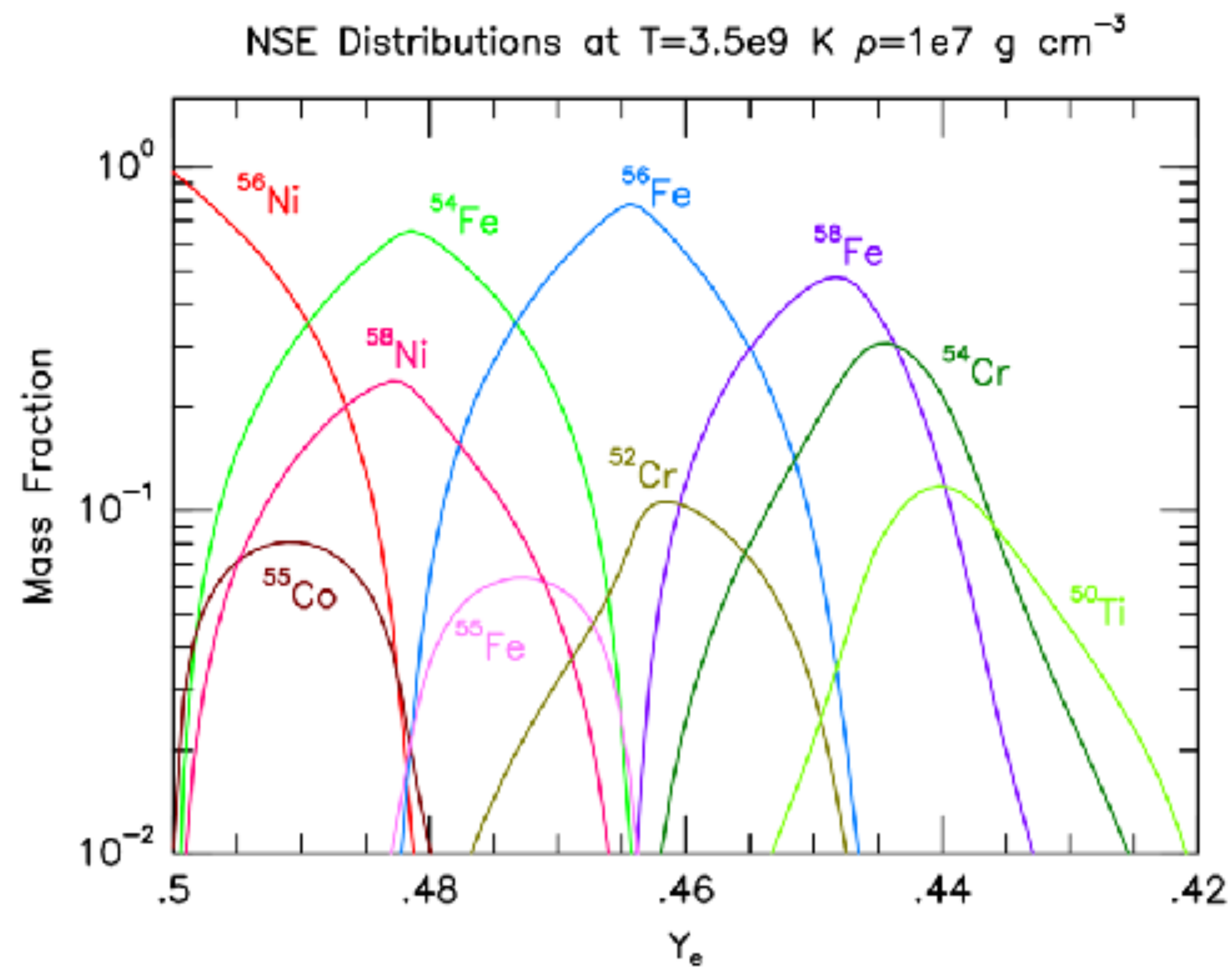
$$\sum_i X_i = 1; \;\; Y_e = \sum \frac{Z_i}{A_i} X_i$$

With the above, each triplet (T, ρ, Y_e) yields a unique solution for (X_i, μ_p, μ_n)

see public code: http://cococubed.asu.edu/code_pages/nse.shtml by F. Timmes and references therein

Reactions in statistical equilibrium





see public code: http://cococubed.asu.edu/code_pages/nse.shtml by F. Timmes and references therein